Image Restoration

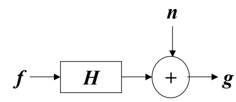
- Image *enhancement* techniques yield ***"better looking"*** images satisfying some subjective criteria.
- Image *restoration* may be defined as *image quality improvement* under *objective* evaluation criteria (least squares, MMSE - minimum mean-squared error) to find the *best possible estimate to the original unknown image from the given degraded image*.
- Restoration requires precise information about the degrading phenomenon, and analysis of the system that produced the degraded image.

Image Restoration

LSI (linear shift-invariant) Degradation Model:

$$g(x,y) = h(x,y) * f(x,y) + n(x,y)$$
$$G(u,v) = H(u,v)F(u,v) + N(u,v)$$

$$g = Hf + n$$



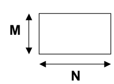
The restoration problem is:

Given g and some knowledge of H, f, and n, find the best possible estimate of f.

Matrix and Vector Representation of Images

- A sampled quantized image may be represented as a matrix or a 2D array of numbers:

$$[f] = \{f_{ij} : i = 1,2,\ldots,M;\ j = 1,2,\ldots,N\}$$



- The image has M rows, each with N elements (N columns).
- Matrix methods may then be used in the analysis of images.
- However, images are not merely arrays of numbers certain constraints are imposed on the image matrix due to the physical properties of the image.

(Reference: E.L. Hall, "Computer Image Processing and Recognition", Academic Press, New York, 1979.)

Matrix and Vector Representation of Images

1. Nonnegativity and upper bound:

$$0 \le f_{ij} \le B$$

2. Finite energy:

$$E = \sum_{i=1}^{M} \sum_{j=1}^{N} f_{ij}^2 \le E_0$$

3. Smoothness:

$$f_{ij} - \frac{1}{8}\begin{pmatrix} f_i - 1, j - 1 & + f_i - 1, j & + f_i - 1, j + 1 \\ + f_i, j - 1 & & + f_i, j + 1 \\ + f_i + 1, j - 1 & + f_i + 1, j & + f_i + 1, j + 1 \end{pmatrix} \le S$$

Matrix and Vector Representation of Images

The image matrix may be converted to a vector by "row ordering":

$$f = [f^1 f^2 \ldots f^M]' \qquad (MN \times 1)$$

where f^i = [f(i,1) f(i,2). . . f(i,N)]' is the i^{th} row vector.

Column ordering may also be performed.

Energy $E = f' f = \sum_{i=1}^{MN} f_i^2$ **(inner product)**

Matrix and Vector Representation of Images

If the image elements are considered to be random variables, the image may be seen as a sample of a stochastic process, and characterized by:

mean $\bar{f} = E\{f\} \qquad (MN.1)$

covariance matrix $E\{(f - \bar{f})(f - \bar{f})'\} \qquad (MN \times MN)$

correlation matrix $E\{f\ f'\} \qquad (MN \times MN)$

***E* { }: *statistical expectation* (average) operator.**

Matrix Representation of Linear Systems Relationship

Considering the 1D linear shift-invariant system for simplicity, we have the input-output relationship given by the convolution integral.

$$g(t) = \int_a^b f(\tau)h(t-\tau)d\tau$$

- **The limits depend upon causality, the nature of h (IIR, FIR), and whether the convolution desired is linear or circular.**
- **While *causality* is an inherent property of physical 1D signal processing systems, *it is not always relevant in the 2D* case as *blurring typically occurs in all directions*.**
- **The limits also depend on the reference origin chosen, whether at the sample or at the center of the signal.**

Matrix Representation of Linear Systems Relationship

1. IIR (Infinite Impulse Response)

Consider the systems to be *causal*, with input starting at t = 0.

$$g(t) = \int_0^t f(\tau)h(t-\tau)d\tau$$

$$g(k) = \sum_{j=0}^{k} f(j)h(k-j)$$

Matrix Representation of Linear Systems Relationship

2. FIR (Finite Impulse Response, Non-causal)

$$g(t) = \int_{t-T/2}^{t+T/2} f(\tau)h(t-\tau)d\tau$$

$$g(k) = \sum_{j=k-M/2}^{k+M/2} f(j)h(k-j)$$

$$g(k) = \sum_{j=0}^{M} f(j+k-M/2)h(M/2-j)$$ moving window

Matrix Representation of Linear Systems Relationships

3. Periodic or Circular Convolution

$$g_P(t) = \int_0^T f(\tau)h(t-\tau)d\tau$$

***T* > (*T_1* + *T_2*) to avoid wrap-around errors;**
***T*, *T_1*, and *T_2* are the durations of *g*, *f*, and *h*, respectively;**
subscript *p* indicates periodic versions of the signals.

$$g_p(k) = \sum_{j=0}^{M-1} f_p(j)h_p(k-j)$$

Matrix Representation of Linear Systems Relationships

Convolution as Matrix Operation

1. IIR $g = Hf$ $\qquad g(k) = \sum_{j=0}^{k} f(j)h(k-j)$

$$\begin{bmatrix} g(0) \\ g(1) \\ g(2) \\ \vdots \\ g(M) \\ \vdots \\ g(N) \end{bmatrix} = \begin{bmatrix} h(0) & & & & \\ h(1) & h(0) & & & \\ h(2) & h(1) & h(0) & & \\ \vdots & & & & \\ h(M) & \dots & & & \\ \vdots & & & & \\ h(N) & \dots & & \dots & h(0) \end{bmatrix} \cdot \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \\ f(M) \\ \vdots \\ f(N) \end{bmatrix}$$

H is Toeplitz-like. There will be zeros in the lower-left portion of H if h has fewer samples than f and g: H is then said to be banded.

Matrix Representation of Linear Systems Relationships

Convolution as Matrix Operation

2. FIR $g = Hf$ $\qquad g(k) = \sum_{j=0}^{M} f(j+k-M/2)h(M/2-j)$

$$\begin{bmatrix} g(0) \\ g(1) \\ g(2) \\ \vdots \\ \vdots \\ \vdots \\ g(N) \end{bmatrix} = \begin{bmatrix} h\left(\frac{M}{2}\right) & h\left(\frac{M}{2}-1\right) & \dots & h(0) & \dots & \dots & h\left(-\frac{M}{2}\right) & 0 & \\ & h\left(\frac{M}{2}\right) & \dots & \dots & h(0) & \dots & \dots & h\left(-\frac{M}{2}\right) & \\ & & & \vdots & & & & & \\ & & & \vdots & & & & & \\ & & & \vdots & & & & & \\ & & & \vdots & & & & & \\ & & & h\left(\frac{M}{2}\right) & & & & \dots & h\left(-\frac{M}{2}\right) \end{bmatrix} \cdot \begin{bmatrix} f\left(-\frac{M}{2}\right) \\ f\left(1-\frac{M}{2}\right) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f\left(N+\frac{M}{2}\right) \end{bmatrix}$$

H is banded and Toeplitz-like.

Each row (except the first) is a right-shift of the previous row.

Matrix Representation of Linear Systems Relationships

3. Periodic or Circular Convolution

$$g_p(k)=\sum_{j=0}^{M-1} f_p(j)h_p(k-j)$$

$$g_p(0) = f_p(0)h_p(1) + f_p(1)h_p(-1) + ... + f_p(M-1)h_p(-M+1),$$

But

and $h_p(-k) = h(M-k)$ by periodicity.

.

$$g_p(0) = f_p(0)h_p(0) + f_p(1)h_p(M-1) + ... + f_p(M-1)h_p(1)$$

and so on.

$$g_p = H_p f_p$$

Matrix Representation of Linear Systems Relationships

$$g_p(k)=\sum_{j=0}^{M-1} f_p(j)h_p(k-j)$$

$$\begin{bmatrix} g_p(0) \\ g_p(1) \\ \vdots \\ g_p(M-1) \end{bmatrix} = \begin{bmatrix} h_p(0) & h_p(M-1) & h_p(M-2) & \cdots & h_p(1) \\ h_p(1) & h_p(0) & h_p(M-1) & \cdots & h_p(2) \\ \vdots & & & & \\ h_p(M-1) & h_p(M-1) & h_p(M-2) & \cdots & h_p(0) \end{bmatrix} \begin{bmatrix} f_p(0) \\ f_p(1) \\ \vdots \\ f_p(M-1) \end{bmatrix}$$

- **Each row of H_p is a right-shift (circular-shift) of the previous row.**
- **H_p is square.**
- **H_p is a circulant matrix.**
- **An important property of a circulant matrix is that it is *diagonalized by the* DFT.**

Matrix Representation of Linear Systems Relationships

Consider the general circulant matrix

$$C = \begin{bmatrix} C(0) & C(1) & C(2) & \cdots & C(N-1) \\ C(N-1) & C(0) & C(1) & \cdots & C(N-2) \\ \vdots & & & & \\ C(1) & C(2) & C(3) & \cdots & C(0) \end{bmatrix}$$

Let $W = \exp\left(i\frac{2\pi}{N}\right), \qquad i=\sqrt{-1}$

Then, W^k, k = 0, 1, 2,…,N - 1, are the N distinet roots of unity, as W^{kn} = 1.

Now consider

$$\lambda(k) = C(0) + C(1)W^k + C(2)W^{2k} + ... + C(N-1)W^{(N-1)k}.$$

Matrix Representation of Linear Systems Relationships

$$\lambda(k) = C(0) + C(1)W^k + C(2)W^{2k} + ... + C(N-1)W^{(N-1)k}.$$

$$\begin{aligned} \lambda(k)W^k &= C(N-1) + C(0)W^k + C(1)W^{2k} + \cdots + C(N-2)W(N-1)^k \\ \lambda(k)W^{2k} &= C(N-2) + C(N-1)W^k + C(0)W^{2k} + \cdots + C(N-3)W^{(N-1)^k} \\ &\vdots \\ \lambda(k)W^{(N-1)^k} &= C(1) + C(2)W^k + C(3)W^{2k} + \cdots + C(0)W^{(N-1)^k}, \end{aligned}$$

i.e., λ(k)W(k) = CW(k),

were W(k) = [1W^k W^{2k}…$W^{(N-1)}$]'.

- **Thus λ(k) is an *eigenvalue* and W(k) is an *eingenvector* of the circulant matrix C.**
- **Since there are N values W^k, k = 0, 1,…N-1, that are distinct, there are N distinct eigenvectors W(k), which may be written as tha N x N matrix** $W = [W(0) \quad W(1)..W(N-1)]$ **that is related to the DFT.**

Matrix Representation of Linear Systems Relationships

- **The eigenvalue relationship may be written as:**

  $$W\Lambda = CW$$

  where all the terms are N X N matrices, and Λ is a diagonal matrix whose terms are equal to λ(κ), κ=-0,1,…, N-!.

  $$C = W\Lambda W^{-1}$$

- **Thus a circulant matrix is a diagonalized by the DFT matrix W.**
- **Returning to periodic convolution, since H_p is circulant, we have**

  $$H_p = WDW^{-1} \quad and \quad g_p = WDW^{-1}f_p$$

Matrix Representation of Linear Systems Relationships

Interpretation:

- **$W^{-1} f_p$ is the DFT of f_p ;**
- **multiplication of this by D corresponds to point-by-point transform-domain filtering whit the DFT of h;**
- **W corresponds to the inverse DFT.**

Classification:

$$F(k) = \frac{1}{N}\sum_{j=0}^{N-1} f_p(j)\exp\left(-i\frac{2\pi kj}{N}\right)$$

$$G(k) = \frac{1}{N}\sum_{j=0}^{N-1} g_p(j)\exp\left(-i\frac{2\pi kj}{N}\right)$$

k = 0, 1,…,N - 1, are the DFTs of f_p and g_p;

Matrix Representation of Linear Systems Relationships

We defined the eigenvalues of the circulant matrix using the first row of H_p, i.e. $h_p(-j)$. Thus the diagonal elements are:

$$D_{kk} = \sum_{j=0}^{N-1} h_p(-j) \exp\left(i \frac{2\pi kj}{N} \right).$$

Since h_p is periodic, summation from 0 to - (N-1) is equal to summation from 0 to (N-1). Thus -j may be replace by j:

$$D_{kk} = \sum_{j=0}^{N-1} h_p(j) \exp\left(- i \frac{2\pi kj}{N} \right).$$

Let the DFT of $h_p(j)$ be $H(k) = \frac{D_{kk}}{N}.$

Matrix Representation of Linear Systems Relationships

The frequency-domain representation of circular convolution is

$$G(k) = N\, H(k)\, F(k),$$

which may be evaluated rapidly using the FFT.

It could further be shown that 2D periodic convolution may be represented by a block-circulant matrix, which is diagonalized by the 2D DFT.

Matrix Representation of Linear Systems Relationships

Block-Circulant Matrices

For two digitized images f(x,y) and h(x,y) of size AxB and CxD, respectively, extended images of size MxN may be formed by padding the functions with zero.

$$f_e(x,y) = \begin{cases} f(x,y) & 0 \le x \le A-1 \quad and \quad 0 \le y \le B-1 \\ 0 & A \le x \le N-1 \quad or \quad B \le y \le M-1 \end{cases}$$

and

$$h_e(x,y) = \begin{cases} h(x,y) & 0 \le x \le C-1 \quad and \quad 0 \le y \le D-1 \\ 0 & C \le x \le N-1 \quad or \quad D \le y \le M-1 \end{cases}$$

The extended functions $f_e(x,y)$ and $h_e(x,y)$ are periodic functions in 2D with M and N in the x and y directions.

Matrix Representation of Linear Systems Relationships

- The convolution of the two functions is given by:

$$g_e(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_e(m,n) h_e(x-m, y-n),$$

  for x = 0,1, 2,…,M-1, and y = 0,1, 2,…, N - 1.
- The result is periodic with the same period (M x N) as of $f_e(x,y)$ and $h_e(x,y)$.
- Overlap of the individual convolution periods is avoided by choosing $M \ge$ (A+C-1) and $N \ge$ (B+D-1).
- The complete discrete degradation model is given by

$$g_e(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_e(m,n) h_e(x-m, y-n) + \eta_e(x,y),$$

  where $\eta_e(x,y)$ is an M X N extended discrete noise image.

Matrix Representation of Linear Systems Relationships

- Let f, g, and n be MN-dimensional vectors formed by attacking the rows of the M x N functions $f_e(x,y)$, $g_e(x,y)$, and $\eta_e(x,y)$.
- Now, the degradation model may be written as

$$g = Hf + n$$

  where f, g; and n are of dimension MN x 1, and H is of dimension MN x MN.

$$H = \begin{bmatrix} H_0 & H_{M-1} & H_{M-2} & \cdots & H_1 \\ H_1 & H_0 & H_{M-1} & \cdots & H_2 \\ H_2 & H_1 & H_0 & \cdots & H_3 \\ \vdots & & & & \vdots \\ H_{M-1} & H_{M-2} & H_{M-3} & \cdots & H_0 \end{bmatrix}$$

Matrix Representation of Linear Systems Relationships

$$H_j = \begin{bmatrix} h_e(j,0) & h_e(j,N-2) & h_e(j,N-2) & \dots & h_e(j,1) \\ h_e(j,1) & h_e(j,0) & h_e(j,N-1) & \cdots & h_e(j,2) \\ h_e(j,2) & h_e(j,1) & h_e(j,0) & \dots & h_e(j,3) \\ \vdots & & & & \vdots \\ h_e(j,N-1) & h_e(j,N-1) & h_e(j,N-3) & \dots & h_e(j,0) \end{bmatrix}$$

H_j is a circulant matrix, and the blocks of H are subscripted in a circular manner; H is a block-circulant matrix.

Matrix Representation of Linear Systems Relationships

- The degradation model expression looks simple.
- However, a direct solution of this expression to obtain f is a monumental processing task for images of practical size.
- For example, if M = N = 512, H is of size 262,144 x 264,144.
- To obtain f directly would require the solution of a system of 262,144 simultaneous linear equations.
- Fortunately, the complexity of this problem can be reduced considerably by taking advantage of the circulant properties of H.

Matrix Representation of Linear Systems Relationships

Diagonalization of block-circulant Matrices

Let

$$\omega_M(i,m) = \exp\left[j\frac{2\pi}{M}im\right]$$

$$\omega_N(k,n) = \exp\left[j\frac{2\pi}{N}kn\right]$$

Define a Matrix W of size MN x MN,
containing M^2 partitions of size N x N.
The imth partition of W is

$$W(i,m) = \omega_M(i,m)W_N$$

For i,m = 0,1, 2,…,M - 1.

Matrix Representation of Linear Systems Relationships

W_N is an N X N matrix with elements

$$W_n(k,n) = \omega_n(k,n)$$

for k,n = 0, 1, 2,…, N-1.
The inverse matrix W^{-1} is also of MN x MN
With W^2 partitions of size N x N.
The imth partitions of W^{-1}, symbolized as $W^{-1}(i,m)$, is

$$W^{-1}(i,m) = \frac{1}{M}\omega_M^{-1}(i,m)W_N^{-1},$$

$$\omega_M^{-1}(i,m) = \exp\left[-j\frac{2\pi}{M}im\right]$$

for i,m = 0, 1, 2,…, M-1.

Matrix Representation of Linear Systems Relationships

The matrix W_N^{-1} has elements

$$W_N^{-1}(k,n) = \frac{1}{N}\omega_N^{-1}(k,n),$$

$$\omega_N^{-1} = \exp\left[-j\frac{2\pi}{N}kn\right]$$

for k,n = 0, 1, 2,…, N - 1.
Direct substitution of elements of W and W^{-1} shows that

$$WW^{-1} = W^{-1}W = \mathrm{I}$$

Where I is the MN x MN identity matrix.

Matrix Representation of Linear Systems Relationships

If H is a block-circulant matrtix, it can de show that

$$H = WDW^{-1}$$

or

$$D = W^{-1}HW$$

where D is a diagonal matrix whose elements D(k,k) are related to the DFT of $h_e(x,y)$.

the transpose of H is.

$$H' = WD^{*}W^{-1}.$$

Orthogonal Functions and Transforms

In signal analysis, it is often useful to represent a signal x(t) over the t_0 to t_0 + T by an expansion of the form.

$$x(t) = \sum_{m=0}^{\infty} a_m \varnothing_m(t).$$

Where the functions $\varnothing$ m(t) are *mutually orthogonal*, i.e.,

$$\int_{t_0}^{t_0+T} \varnothing_m(t)\varnothing_n^*(t)dt = \begin{cases} C & m = n \\ 0 & m \neq n. \end{cases}$$

if C = 1 the functions are orthonormal.

Orthogonal Functions and Transforms

The coefficients a_m may

$$a_m = \frac{1}{C}\int_{t_0}^{t_0+T} x(t)\varnothing_m^*(t)\,dt\ ,\ m = 0,1,2,...,$$

i.e., a_m is the projection of x(t) on to$\varnothing_m$(t).

The set {$\varnothing_m$(t)} is said to be complete or closed if there exists no square-integrable function x(t) for which

$$\int_{t_0}^{t_0+T} x(t)\varnothing_m^*(t)\,dt = 0,\quad m = 0,1,2,...$$

If this is true, x(t) should be a member of the set

When the set {$\varnothing_m$(t)} is complete, it is said to be an orthogonal basis, and may be used for accurate representation of signals, e.g., the Fourier series

Note: x(t) and the $\varnothing_m$(t)'s must be square-integrable.

Orthogonal Functions and Transforms

With the signal or image expressed as an MN x 1 vector or column matrix, we may consider representation of transformations using MN x MN orthogonal matrices:

$$L\ L^{*\prime} = 1$$
$$F = L\ and\quad f = L^{*\prime}\ F$$

representing

$$F_i = \sum_{j=1}^{MN} L_{ij}\, f_j\ and\quad f_i = \sum_{j=1}^{MN} L^*_{ji} F_j\ ,$$

i = 1, 2,…,MN.

For images of size M x N. the transformation matrices will be of size MN x MN, leading to computational difficulties.

Orthogonal Functions and Transforms

General representation of image transforms:

$$F(k,l) = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f(m,n)\, g(m,n,k,l).$$

$$f(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{t=0}^{N-1} F(k,l)\, h(m,n,k,l),$$

where g(m, n, k, l) is the forward transform kernel and h(m, n, k, l) is the inverse transform kernel.

The kernel is said to be separable if g(m, n, k, l) = g_1 (m, k) 9_2 (n, l), and symmetric in addition if 9_1 and g_2 are functionally equal.

Then, the 2D transform may be computed in two simpler steps: ID row transforms followed by 1D column transforms.

Orthogonal Functions and Transforms

$$F_1(m,l) = \sum_{n=0}^{N-1} f(m,n)\, g(n,l)\quad m,l = 0,1,...\, N-1,$$

$$F(k,l) = \sum_{m=0}^{N-1} F_1(m,l)\, g(m,k)\quad k,l = 0,1,...,\ N-1.$$

The 2D Fourier transform kernel

$$\exp[-j2\pi(mk+nl)/N] = \exp[-2j\pi mk/N]\exp[-2j\pi nl/N]$$

is separable and symmetric.

Orthogonal Functions and Transforms

The 2DDFT may be written as $F = \frac{1}{N} WfW$

where f is the NxN image matrix, and W is a symmetric NxN matrix with $W_{km} = \exp[-j2\pi km/N]$ **(only N distict values).**

$$\begin{bmatrix}
W_0 & W_0 & W_0 & W_0 & W_0 & W_0 & W_0 & W_0 \\
W_0 & W_1 & W_2 & W_3 & W_4 & W_5 & W_6 & W_7 \\
W_0 & W_2 & W_4 & W_6 & W_0 & W_2 & W_4 & W_6 \\
W_0 & W_3 & W_6 & W_1 & W_4 & W_7 & W_2 & W_5 \\
W_0 & W_4 & W_0 & W_0 & W_0 & W_4 & W_0 & W_4 \\
W_0 & W_5 & W_2 & W_2 & W_4 & W_1 & W_6 & W_3 \\
W_0 & W_6 & W_4 & W_4 & W_0 & W_6 & W_4 & W_2 \\
W_0 & W_7 & W_6 & W_6 & W_4 & W_3 & W_2 & W_1
\end{bmatrix}$$

Orthogonal Functions and Transforms

The DFT matrix is symmetric and unitary:

$$\sum_{m=0}^{N-1} W_{mk} W^*_{ml} = \begin{cases} N\ k = 1 \\ 0\ k \neq 1 \end{cases}$$

I.e., the rows/columns are mutually orthogonal

Then, $W^{-1} = \frac{1}{N} W^*\ and\quad f = \frac{1}{N} W^* FW^*.$

A number of transforms such as the Fourier, Walsh, Hadamard, and Discrete Cosine may be expressed as F = A f A.

The transform matrices may be decomposed into products of matrices with fewer nonzero elements, reducing redundancy and computational requirements.

The DFT matrix may be factored into a product of 2 in N sparse and diagonal matrices, which may be considered to be the basis of the FFT algorithm

Orthogonal Functions and Transforms

The Walsh-Hadamard Transform

The orthogonal, complete set of walsh functions defined over interval $0 \le x \le 1$ is given by the iterative relationship (in 1D);

$$\varnothing_0(x) = 1; \qquad \varnothing_1 = \begin{cases} 1 & x < 1/2 \\ -1 & x \ge 1/2, \end{cases}$$

$$\varnothing_n(x) = \begin{cases} \varnothing_{[n/2]}(2x) & x < 1/2 \\ \varnothing_{[n/2]}(2x-1) & x \ge 1/2, \quad n\, odd \\ -\varnothing_{[n/2]}(2x-1) & x \ge 1/2, \quad n\, even, \end{cases}$$

where [n/2]is the integral part of n/2.

Orthogonal Functions and Transforms

ϕ_n iS generated by compression of $\phi_{[n/2]}$ into its first half and $\pm\phi_{[n/2]}$ into its second half, and is even/odd as n.

To generate discrete Walsh functions, the number of samples (equispaced) should be 2^n to satisfy the above requirement.

Walsh functions are ordered by the number of zero-crossings in the interval (0,1), called sequency.

If the Walsh functions with the number of zero-crossings $\le (2^n - 1)$ are sampled with $N = 2^n$ uniformly-spaced points, we get a square matrix representation, which is orthogonal with rows ordered with increasing number of zero-crossings.

Orthogonal Functions and Transforms

For N = 8:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

The major advantage of the Walsh transform is that the kernel has integers with values +1 and -1 only, i.e., the transform involves only addition and subtraction of the image pixels.

Orthogonal Functions and Transforms

Except for the ordering of rows, discrete Walsh matrices are equivalent to Hadamard matrices of rank 2^n, which are easily.

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}; \qquad H_{2n} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}$$

Then, letting $H = \frac{1}{\sqrt{N}} H_N$ **, the Walsh-Hadamard transform may be expressed as**

$$\boxed{F = HfH, \qquad f = HFH}$$

applications: image coding, sequency filtering, pattern recognition.

Orthogonal Functions and Transforms

Also known as the Principal Component, Hotelling transform, or the Eigenvector transform (Ref: Hall).

This transform is based on statistical properties of the given image, which i3 treated as a random vector X.

Mean vector $\mu = E\{X\} = \int X\, p(X)\, dX$

Covariance matrix: $$\sum = E\{(X-\mu)'\} = E\{X, X'\} = \mu\mu$$
$$= \begin{bmatrix} \sigma_{11}^2 & \sigma_{12} \dots & \sigma_{1n} \\ \vdots & \vdots & \vdots \\ \sigma_{n1} & \sigma_{n2} \dots & \sigma_{nn}^2 \end{bmatrix}$$

$\sigma_{ij} = E\{(x_i - \mu_I)(x_j - \mu_j)\}$; $\mu_j = E\{x_i\}$; I,j = 1, 2,…,n.

Orthogonal Functions and Transforms

NOTE:

Two random vectors X_i and X_j are

- **Uncorrelated if $E\{X'_I X_j\} = E\{X'_I\} E\{X_j\}$**

(then Σ is diagonal and R is the identity matrix}

- **Orthogonal if $E\{X'_I X_j\} = 0$**

(if $E\{X'_I\} = 0$ or $E\{X'_I\} = 0$, orthogonal = uncorrelated)

- **Statistically independent if $p(X_i, X_j) = p(X_i)p(X_j)$**

(then X_i and X_j are uncorrelated).

Orthogonal Functions and Transforms

A random vector X may be represented without error by deterministic transformation of the form:

$$X = AY = \sum_{i=1}^{n} y_i A_i$$

where A = $[A_1\ A_2 \ldots A_n]$, $|A| \neq 0$.

The matrix A may be considered to be made up of n.

linearly-independent column vectors, called the basis vector which span n-dimensional space containing X.

Let A be orthogonal, i.e.,

$$A_i^{'} A_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

if follows that $A'A = I$ or $A^{-1} = A'$.

Orthogonal Functions and Transforms

Then; Y $A'X = \Sigma^n_{i=1} A'_i x_i$.

Each component of Y contributes to the representation of X.

Suppose we wish to the use m<n components of Y.

The omitted components of Y may be replace with other values b_i,i = m+1, . . . ,n:

$$X = \sum_{i=1}^{m} y_i A_i + \sum_{i=m+1}^{n} b_1 A_i .$$

Error $\varepsilon = X - \hat{X} = \sum_{i=m+1}^{n} (y_i - b_i) A_i .$

Orthogonal Functions and Transforms

The components of Y are mutually uncorrelated:

$$\sum_Y = A' \sum_X A = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & . & & \\ & & & . & \\ & & & & \lambda_n \end{bmatrix} = \Lambda .$$

Since the eigenvalues λi are equal to the variances of y_i, selection of the largest eigenvalues implies selection of maximum-variance (information content) components.

Applications: Image coding and compression, feature extraction.

Difficulties:In computing the eigenvectors/values of large covariance matrices.

A transformation is valid only for the corresponding set of images.

